

A BEALE-KATO-MAJDA BLOW-UP CRITERION FOR THE 3-D COMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. We prove a blow-up criterion in terms of the upper bound of the density for the strong solution to the 3-D compressible Navier-Stokes equations. The initial vacuum is allowed. The main ingredient of the proof is *a priori* estimate for an important quantity under the assumption that the density is upper bounded, whose divergence can be viewed as the effective viscous flux.

1. INTRODUCTION

In this paper, we consider the isentropic compressible Navier-Stokes system in three dimensional space. The system reads

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & \text{in } (0, T) \times \Omega, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - Lu + \nabla p = 0, & \text{in } (0, T) \times \Omega, \end{cases}$$

together with the initial-boundary conditions

$$(1.2) \quad (\rho(t, x), u(t, x))|_{t=0} = (\rho_0(x), u_0(x)), \quad \text{in } \Omega,$$

$$(1.3) \quad u(t, x) = 0, \quad \text{on } (0, T) \times \partial\Omega.$$

Here Ω is either \mathbb{R}^3 or a bounded domain in \mathbb{R}^3 , ρ and u are the density and velocity of the fluid respectively, $p = a\rho^\gamma$ with $\gamma > 1$ is the pressure. The Lamé operator L is defined by

$$Lu = \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u,$$

with constant viscosity coefficients μ and λ satisfying

$$(1.4) \quad \mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

In the absence of vacuum for the initial density, the local existence of strong solution as well as the global existence of strong solution and weak solution with the initial data close to an equilibrium state were well developed, see [21, 22, 24, 14, 10, 6] and references therein. The global existence of weak solution for large initial data was first solved by P. L. Lions in [20] for $\gamma \geq \frac{9}{5}$. E. Feireisl, A. Novotný and H. Petzeltová [13] extended Lions's

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result to the case of $\gamma > \frac{3}{2}$. S. Jiang and P. Zhang [18, 19] proved the global existence of weak solution for any $\gamma > 1$ for the spherically symmetric or axisymmetric initial data. However, the regularity and uniqueness of weak solutions are completely open even in the case of two dimensional space. The only known result is the work of Kazhikhov and Vaĭgant [27], where they proved the global existence of strong solution for the system (1.1) in $\Omega = \mathbb{T}^2$ under the assumption that μ is a constant and $\lambda = \rho^\beta$ with $\beta > 3$. On the other hand, when the initial density is compactly supported, Z. Xin [28] proved that smooth solution will blow up in finite time in the whole space.

To proceed we introduce some notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$D^{k,r}(\Omega) \stackrel{\text{def}}{=} \{u \in L^1_{loc}(\Omega) : \|\nabla^k u\|_{L^r(\Omega)} < \infty\},$$

$$W^{k,r}(\Omega) \stackrel{\text{def}}{=} L^r(\Omega) \cap D^{k,r}(\Omega), H^k(\Omega) = W^{k,2}(\Omega), D^k(\Omega) = D^{k,2}(\Omega),$$

$$D^1_0(\Omega) \stackrel{\text{def}}{=} \{u \in L^6(\Omega) : \|\nabla u\|_{L^2(\Omega)} < \infty \text{ and } u = 0 \text{ on } \partial\Omega\},$$

$$H^1_0(\Omega) \stackrel{\text{def}}{=} L^2(\Omega) \cap D^1_0(\Omega), \|u\|_{D^{k,r}(\Omega)} = \|\nabla^k u\|_{L^r(\Omega)}.$$

When the initial vacuum is allowed, the local well-posedness and blow-up criterion for strong solutions to the compressible Navier-Stokes equations were established in a series of papers [7, 8, 9] by Cho, Choe and Kim. Here we write down one of those results.

Theorem 1.1. Let Ω be a bounded smooth domain or \mathbb{R}^3 and $q \in (3, 6]$. Suppose that $\rho_0 \geq 0$ and belongs to $W^{1,q}(\Omega) \cap H^1(\Omega) \cap L^1(\Omega)$, $u_0 \in D^1_0(\Omega) \cap D^2(\Omega)$ with the following compatibility condition satisfied,

$$(1.5) \quad -\mu \Delta u_0 - (\mu + \lambda) \nabla \operatorname{div} u_0 + \nabla p(\rho_0) = \sqrt{\rho_0} g,$$

for some vector field $g \in L^2(\Omega)$. Then there exist a time $T \in (0, \infty]$ and a unique strong solution (ρ, u) to (1.1) such that

$$\rho \in C([0, T], H^1 \cap W^{1,q}(\Omega)), u \in C([0, T], D^2(\Omega)) \cap L^2(0, T; D^{2,q}(\Omega)).$$

Moreover, let T^* be a maximal existence time of the solution. If $T^* < \infty$, then there holds

$$(1.6) \quad \limsup_{t \uparrow T^*} (\|\rho(t)\|_{W^{1,q}(\Omega)} + \|u(t)\|_{D^1_0(\Omega)}) = \infty.$$

Since the initial vacuum is allowed, it is then important to investigate the possible blow-up mechanism of the solution. In their recent works [15, 16], X. Huang and Z. Xin established a Beale-Kato-Majda blow up criterion for the above strong solution. More precisely,

Theorem 1.2. Assume that the coefficients of the operator L satisfies (1.4) and moreover,

$$(1.7) \quad \lambda < 7\mu.$$

Let (ρ, u) be the strong solution constructed in Theorem 1.1 and T^* be a maximal existence time. If $T^* < \infty$, then

$$(1.8) \quad \lim_{T \uparrow T^*} \|\nabla u\|_{L^1(0, T; L^\infty(\Omega))} = \infty.$$

Recently, J. Fan, S. Jiang and Y. Ou [12] also obtained a similar result for the compressible heat-conductive flows. On the other hand, for the 2D compressible Navier-Stokes equations in \mathbb{T}^2 , B. Desjardins [11] proved more regularity of weak solution under the assumption that the density is upper bounded; Very recently, L. Jiang and Y. Wang [17], Y. Sun and Z. Zhang [26] obtained a blow-up criterion in terms of the upper bound of the density for the strong solution. In [26], the initial vacuum is allowed and the domain includes the bounded domain. Note that the $L^1(0, T; L^\infty(\Omega))$ bound for ∇u immediately implies the upper bound for the density ρ .

The purpose of this paper is to obtain a Beale-Kato-Majda blow-up criterion in terms of the upper bound of the density for the 3-D compressible Navier-Stokes equations. Our main result is stated as follows.

Theorem 1.3. Assume that (ρ, u) is the strong solution constructed in Theorem 1.1. Let μ, λ be as in Theorem 1.2 and T^* be a maximal existence time of the solution. If $T^* < \infty$, then we have

$$(1.9) \quad \limsup_{T \uparrow T^*} \|\rho(t)\|_{L^\infty(0, T; L^\infty(\Omega))} = \infty.$$

Remark 1.4. This result seems surprise, if we compare with the incompressible Navier-Stokes equations where the density is a constant. It is well-known that if we have some kind of control for the pressure, the Leray weak solution is in fact smooth for the incompressible Navier-Stokes equations, see [4]. For the compressible Navier-Stokes equations, the pressure is determined by the density, the bound of the density thus implies a bound for the pressure. From this viewpoint, our result seems natural.

Remark 1.5. In a forthcoming paper, we will extend similar result to the compressible heat-conductive flows.

Let us conclude this section by introducing the main idea of our proof. First of all, if the density is upper bounded, we can obtain a high integrability of the velocity, see Lemma 3.1. This bound can be used to control the nonlinear term. The trouble is to control the density, which satisfies a transport equation. In order to propagate the regularity of the density, it is necessary to require that the velocity is bounded in $L^1(0, T; W^{1, \infty}(\Omega))$. On

the other hand, we have to obtain a priori bound of $\nabla \rho$ in order to prove $u \in L^1(0, T; W^{1,\infty}(\Omega))$. To overcome this difficulty, we introduce an important quantity w defined by $w = u - v$, where v is the solution of Lamé system

$$\begin{cases} \mu \Delta v + (\lambda + \mu) \nabla \operatorname{div} v = \nabla p(\rho) & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case of $\Omega = \mathbb{R}^3$, $(\lambda + 2\mu) \operatorname{div} w = (\lambda + 2\mu) \operatorname{div} u - p \stackrel{\text{def}}{=} G$. It is well known that G is called the effective viscous flux, which plays an important role in the existence theory of weak solution. A key point is that we can obtain the better regularity of w than u under the only assumption that the density is upper bounded. More precisely, we proved that $\nabla^2 w \in L^2(0, T; L^6(\Omega))$, which combined with the bound of the density implies that $\nabla u \in L^2(0, T; L^\infty(\Omega) + L^\infty(0, T; BMO(\Omega)))$. This bound still does not imply that ∇u is bounded in $L^1(0, T; L^\infty(\Omega))$. We need to introduce the second key ingredient: a logarithmic estimate for Lamé system. Then the result can be deduced by combining the above two estimates into the energy estimates for the density.

2. PRELIMINARIES

Consider the following boundary value problem for the Lamé operator L

$$(2.1) \quad \begin{cases} \mu \Delta U + (\mu + \lambda) \nabla \operatorname{div} U = F, & \text{in } \Omega, \\ U(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $U = (U_1, U_2, U_3)$, $F = (F_1, F_2, F_3)$. It is well known that under the assumption (1.4), (2.1) is a strongly elliptic system. If $F \in W^{-1,2}(\Omega)$, then there exists a unique weak solution $U \in D_0^1(\Omega)$. We begin with recalling various estimates for this system in $L^q(\Omega)$ spaces.

Proposition 2.1. Let $q \in (1, \infty)$ and U be a solution of (2.1). There exists a constant C depending only on λ, μ, q and Ω such that the following estimates hold.

(1) If $F \in L^q(\Omega)$, then

$$(2.2) \quad \begin{cases} \|D^2 U\|_{L^q(\mathbb{R}^3)} \leq C \|F\|_{L^q(\mathbb{R}^3)}, \\ \|U\|_{W^{2,q}(\Omega)} \leq C \|F\|_{L^q(\Omega)}; \quad \text{if } \Omega \text{ is a bounded domain.} \end{cases}$$

(2) If $F \in W^{-1,q}(\Omega)$ (i.e., $F = \operatorname{div} f$ with $f = (f_{ij})_{3 \times 3}$, $f_{ij} \in L^q(\Omega)$), then

$$(2.3) \quad \begin{cases} \|DU\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^q(\mathbb{R}^3)}, \\ \|U\|_{W^{1,q}(\Omega)} \leq C \|f\|_{L^q(\Omega)}; \quad \text{if } \Omega \text{ is a bounded domain.} \end{cases}$$

(3) If $F = \operatorname{div} f$ with $f_{ij} = \partial_k h_{ij}^k$ and $h_{ij}^k \in W_0^{1,q}(\Omega)$ for $i, j, k = 1, 2, 3$, then

$$(2.4) \quad \|U\|_{L^q(\Omega)} \leq C\|h\|_{L^q(\Omega)}.$$

Proof. In the case when Ω is a bounded domain, the estimates (2.2) and (2.3) are classical for strongly elliptic systems, see for example [3]. The estimate (2.4) can be proved by a duality argument with the help of (2.2). In the case of $\Omega = \mathbb{R}^3$, one can give an explicit representation formula for the solution as follows. Taking divergence on both sides of (2.1), one finds

$$\operatorname{div} U = \frac{1}{\lambda + 2\mu} \Delta^{-1} \operatorname{div} F.$$

Substituting this into (2.1) gives us

$$\Delta U = \frac{1}{\mu} F - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \nabla \Delta^{-1} \operatorname{div} F.$$

Denote the Riesz transform $R = (R_1, R_2, R_3) = \nabla \Delta^{-1/2}$. Then

$$\Delta U = \frac{1}{\mu} F - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} R(R \cdot F).$$

Hence for $i, j, k = 1, 2, 3$,

$$\partial_{ij} U_k = \frac{1}{\mu} R_i R_j F_k - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} R_i R_j R_k (R \cdot F).$$

The classical $L^q(\mathbb{R}^3)$ -boundedness for Riesz transform gives

$$\|D^2 U\|_{L^q(\mathbb{R}^3)} \leq C(q) \frac{2\lambda + 3\mu}{\mu(\lambda + 2\mu)} \|F\|_{L^q(\mathbb{R}^3)}.$$

Similar argument gives the estimates (2.3) and (2.4). \square

We need an endpoint estimate for L in the case $q = \infty$. Let $BMO(\Omega)$ stand for the John-Nirenberg's space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} \stackrel{\text{def}}{=} \|f\|_{L^2(\Omega)} + [f]_{BMO},$$

with

$$[f]_{BMO(\Omega)} \stackrel{\text{def}}{=} \sup_{x \in \Omega, r \in (0, d)} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| dy,$$

$$f_{\Omega_r(x)} = \int_{\Omega_r(x)} f(y) dy = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy.$$

Here $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center x and radius r and d is the diameter of Ω . $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$. Note that

$$[f]_{BMO(\Omega)} \leq 2\|f\|_{L^\infty(\Omega)}.$$

Proposition 2.2. If $F = \operatorname{div} f$ with $f = (f_{ij})_{3 \times 3}$, $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$, then $\nabla U \in BMO(\Omega)$ and there exists a constant C depending only on λ, μ and Ω such that

$$(2.5) \quad \|\nabla U\|_{BMO(\Omega)} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

Proof. When Ω is a bounded domain, the estimate (2.5) can be found in [1] for a more general setting. Now if $\Omega = \mathbb{R}^3$ we use the representation formula for ∇U . Since

$$\Delta U = \frac{1}{\mu} \operatorname{div} f - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \nabla \Delta^{-1} \operatorname{div} \operatorname{div} f = \frac{1}{\mu} \operatorname{div} f - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \nabla G,$$

with $G = \sum_{i,j=1}^3 R_i R_j f_{ij}$. For $k, l = 1, 2, 3$,

$$\partial_k U_l = \frac{1}{\mu} R_k \sum_{j=1}^3 R_j f_{lj} - \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} R_k R_l G.$$

By the Fefferman-Stein's classical result on BMO -boundedness of singular integral operators [25], there exists an absolute constant $C > 0$ such that

$$[\nabla U]_{BMO(\mathbb{R}^3)} \leq C \frac{2\lambda + 3\mu}{\mu(\lambda + 2\mu)} \|f\|_{L^\infty(\Omega)}.$$

This inequality combined with (2.3) with $q = 2$ yields (2.5). \square

In the next lemma, we will give a variant of the Brezis-Wagner's inequality [5]. To our knowledge, such a kind of inequality was first established in [23] in the case of $\Omega = \mathbb{R}^3$. For the reader's convenience, we will give a proof in the case when Ω is a bounded Lipschitz domain, see also [26].

Lemma 2.3. Let $\Omega = \mathbb{R}^3$ or be a bounded Lipschitz domain and $f \in W^{1,q}(\Omega)$ with $q \in (3, \infty)$. There exists a constant C depending on q and the Lipschitz property of Ω such that

$$(2.6) \quad \|f\|_{L^\infty(\Omega)} \leq C \left(1 + \|f\|_{BMO(\Omega)} \ln(e + \|\nabla f\|_{L^q(\Omega)}) \right).$$

Proof. First note that for a Lipschitz domain, the following so-called A -property holds:

There exist two constants $A \geq 1$ and $r_0 \in (0, d)$ such that for any $r \in (0, r_0)$ and $x \in \Omega$,

$$|\Omega_r(x)| \leq |B_r(x)| \leq A |\Omega_r(x)|.$$

Without loss of generality we assume $r_0 \leq 1$.

First we give an estimate for $|f_{\Omega_r(x)}|$ with $0 < r < r_0$ and $x \in \Omega$. If $r \geq \frac{1}{2}r_0$, then

$$|f_{\Omega_r(x)}| \leq \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y)| dy \leq C \|f\|_{L^2(\Omega)}.$$

If $r < \frac{1}{2}r_0$, then there exists some integer $k \geq 1$ such that

$$\frac{r_0}{2^{k+1}} \leq r < \frac{r_0}{2^k}, \quad k \leq C(1 + |\ln r|).$$

Denoting $\Omega_j = \Omega_{2^j r}(x)$ for $j = 0, 1, \dots, k$, we have

$$\begin{aligned} |f_{\Omega_r(x)}| &\leq \sum_{j=1}^k |f_{\Omega_{j-1}} - f_{\Omega_j}| + |f_{\Omega_k}| \\ &\leq \sum_{j=1}^k \int_{\Omega_{j-1}} |f(y) - f_{\Omega_j}| dy + C \|f\|_{L^2(\Omega)} \\ &\leq 2^N A \sum_{j=1}^k \int_{\Omega_j} |f(y) - f_{\Omega_j}| dy + C \|f\|_{L^2(\Omega)} \\ &\leq Ck[f]_{BMO(\Omega)} + C\|f\|_{L^2(\Omega)} \leq C(1 + |\ln r|)\|f\|_{BMO(\Omega)}. \end{aligned}$$

We conclude that there exists a constant $C = C(A, r_0, N)$ such that

$$|f_{\Omega_r(x)}| \leq C(1 + |\ln r|)\|f\|_{BMO(\Omega)},$$

which together with Sobolev embedding theorem in a Lipschitz domain [2] ensures that for any fixed $x \in \Omega$ and small enough $\varepsilon > 0$ we have

$$|f(x)| \leq |f(x) - f_{\Omega_\varepsilon(x)}| + |f_{\Omega_\varepsilon(x)}| \leq C \left(\varepsilon^{1-\frac{N}{q}} \|f\|_{W^{1,q}(\Omega)} + (1 + |\ln \varepsilon|)\|f\|_{BMO(\Omega)} \right).$$

A suitable choice of ε yields the inequality (2.6). \square

In the subsequent context we will use $L^{-1}F$ to denote the unique solution U of the Lamé system (2.1).

3. A PRIORI ESTIMATES FOR THE EFFECTIVE VISCOUS FLUX

In what follows, we assume that (ρ, u) is a strong solution of (1.1) in $[0, T)$ with the regularity stated in Theorem 1.1.

Standard energy estimates yields that for any $t \in [0, T)$,

$$\begin{aligned} \|\rho(t)\|_{L^1(\Omega)} &\leq \|\rho_0\|_{L^1(\Omega)}, \\ \|\rho(t)\|_{L^\gamma(\Omega)}^\gamma + \|\rho|u|^2(t)\|_{L^1(\Omega)} + \|\nabla u\|_{L^2((0,t)\times\Omega)}^2 \\ &\leq C(\|\rho_0\|_{L^\gamma(\Omega)}^\gamma + \|\rho_0|u_0|^2\|_{L^1(\Omega)}). \end{aligned}$$

Note that by the assumption on ρ_0, u_0 ,

$$\|\rho_0\|_{L^\gamma(\Omega)}^\gamma \leq \|\rho_0\|_{L^\infty(\Omega)}^{\gamma-1} \|\rho_0\|_{L^1(\Omega)}, \quad \|\rho_0|u_0|^2\|_{L^1(\Omega)} \leq \|\rho_0\|_{L^{3/2}(\Omega)} \|u_0\|_{L^6(\Omega)}^2.$$

We thus have the following bounds

$$(3.1) \quad \|\rho\|_{L^\infty(0,T;L^1(\Omega))}, \quad \|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(\Omega))}, \quad \|\nabla u\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

Here C depends only on μ, λ, γ, a and ρ_0, u_0 .

In what follows the dependence of the constant C on μ, λ, γ, a and Ω will not be mentioned.

The following lemma is the first key step, whose argument comes from [14] and [16].

Lemma 3.1. Assume that $\mu < 7\lambda$ and the density ρ satisfies

$$(3.2) \quad \|\rho\|_{L^\infty(0,T;L^\infty(\Omega))} \leq M.$$

There exists $r \in (3, 6)$ such that $\rho|u|^r \in L^\infty(0, T; L^1(\Omega))$ with

$$\|\rho|u|^r\|_{L^\infty(0,T;L^1(\Omega))} \leq C.$$

Here C depends on $T, \|\rho_0\|_{L^\infty(\Omega)}, \|\nabla u_0\|_{L^2(\Omega)}, M$.

Proof. Multiplying the second equation of (1.1) by $r|u|^{r-2}u$, and integrating the resulting equation on Ω to obtain

$$(3.3) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho|u|^r dx + \int_{\Omega} r|u|^{r-2} (\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2) \\ & \quad + r(r-2) (\mu|u|^{r-2}|\nabla|u||^2 + (\lambda + \mu)(\operatorname{div} u)|u|^{r-3}u \cdot \nabla|u|) dx \\ & = \int_{\Omega} rp(\rho) \operatorname{div}(|u|^{r-2}u) dx \end{aligned}$$

By using the fact $|\nabla u| \geq |\nabla|u||$, the term in the second integrand can be estimated from below by

$$\begin{aligned} & r|u|^{r-2} [\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2 + (r-2)\mu|\nabla|u||^2 \\ & \quad - (\lambda + \mu)(r-2)|\nabla|u|||\operatorname{div} u|] \\ & \geq r|u|^{r-2} [\mu|\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u - \frac{r-2}{2}|\nabla|u||)^2 - (\lambda + \mu)\frac{(r-2)^2}{4}|\nabla|u||^2 \\ & \quad + (r-2)\mu|\nabla|u||^2] \\ & \geq r|u|^{r-2} [\mu|\nabla u|^2 + (r-2)(\mu - (\lambda + \mu)\frac{r-2}{4})|\nabla|u||^2] \end{aligned}$$

Recalling that $\lambda < 7\mu$, there exists $r \in (3, 6)$ such that the last term is greater than

$$c|u|^{r-2}|\nabla u|^2.$$

On the other hand, because of $\|\rho\|_{L^\infty} \leq M$, we find that the right-hand side of (3.3) is controlled by

$$C \int_{\Omega} \rho^{\frac{r-2}{2r}} |u|^{r-2} |\nabla u| dx \leq \epsilon \int_{\Omega} |u|^{r-2} |\nabla u|^2 dx + \frac{C}{\epsilon} \left(\int_{\Omega} \rho|u|^r dx \right)^{\frac{r-2}{r}}.$$

Taking $\epsilon = \frac{\epsilon}{2}$ to yield that

$$\frac{d}{dt} \int_{\Omega} \rho |u|^r dx \leq C \left(\int_{\Omega} \rho |u|^r dx \right)^{\frac{r-2}{r}},$$

which together with the following bound

$$\|\rho_0 |u_0|^r\|_{L^1(\Omega)} \leq \|\rho_0\|_{L^{\frac{6}{6-r}}(\Omega)} \|u_0\|_{L^6(\Omega)}^r \leq C \|\rho_0\|_{L^{\frac{6}{6-r}}(\Omega)} \|\nabla u_0\|_{L^2(\Omega)}^r,$$

implies the desired estimate. \square

Now for each $t \in [0, T)$, we denote $v(t, x) \stackrel{\text{def}}{=} L^{-1} \nabla p(\rho)$. That is, $v(t)$ is the solution of

$$(3.4) \quad \begin{cases} \mu \Delta v + (\lambda + \mu) \nabla \operatorname{div} v = \nabla p(\rho) & \text{in } \Omega, \\ v(t, x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to Proposition 2.1, for any $q \in (1, \infty)$, there exists a constant C independent of t such that

$$(3.5) \quad \begin{aligned} \|\nabla v(t)\|_{L^q(\Omega)} &\leq C \|p(\rho(t))\|_{L^q(\Omega)}, \\ \|\nabla^2 v(t)\|_{L^q(\Omega)} &\leq C \|\nabla p(\rho(t))\|_{L^q(\Omega)}. \end{aligned}$$

Now let us introduce an important quantity

$$w = u - v,$$

whose divergence can be viewed as the effective viscous flux.

An important observation is that this quantity possesses more regularity information than u does under the assumption that the density is upper bounded. More precisely,

Proposition 3.2. Under the assumption (3.2), we have

$$(3.6) \quad \|\nabla w\|_{L^\infty(0,T;L^2(\Omega))}, \|\rho^{\frac{1}{2}} \partial_t w\|_{L^2((0,T) \times \Omega)}, \|\nabla^2 w\|_{L^2((0,T) \times \Omega)} \leq C.$$

Here the constant C depends on $\|\rho_0\|_{L^\infty(\Omega)}, \|\nabla u_0\|_{L^2(\Omega)}, M, T$.

Proof. By using the continuity equation, we find that w satisfies

$$(3.7) \quad \begin{cases} \rho \partial_t w - \mu \Delta w - (\lambda + \mu) \nabla \operatorname{div} w = \rho F, & \text{in } (0, T) \times \Omega, \\ w(t, x) = 0 & \text{on } [0, T) \times \partial\Omega, \quad w(0, x) = w_0(x), & \text{in } \Omega, \end{cases}$$

with $w_0(x) = u_0(x) + v_0(x)$ and

$$\begin{aligned} F &= -u \cdot \nabla u - L^{-1} \nabla(\partial_t p(\rho)) \\ &= -u \cdot \nabla u + L^{-1} \nabla \operatorname{div}[p(\rho)u] + L^{-1} \nabla[(\rho p'(\rho) - p(\rho)) \operatorname{div} u]. \end{aligned}$$

Multiplying the first equation of (3.7) by $\partial_t w$ and integrating the resulting equation over Ω to obtain ,

$$\frac{d}{dt} \int_{\Omega} \mu |\nabla w|^2 + (\lambda + \mu) |\operatorname{div} w|^2 dx + \int_{\Omega} \rho |\partial_t w|^2 dx = \int_{\Omega} \rho F \cdot \partial_t w dx,$$

which together with Hölder inequality and Young's inequality gives

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \mu |\nabla w|^2 + (\lambda + \mu) |\operatorname{div} w|^2 dx + \frac{1}{2} \int_{\Omega} \rho |\partial_t w|^2 dx \\ (3.8) \quad & \leq \frac{1}{2} \|\sqrt{\rho} F\|_{L^2(\Omega)}^2. \end{aligned}$$

Now let us estimate $\|\sqrt{\rho} F\|_{L^2(\Omega)}^2$. We get by Lemma 3.1 and (3.5) that

$$\begin{aligned} \|\sqrt{\rho} u \cdot \nabla u\|_{L^2(\Omega)} & \leq C \|\rho^{\frac{1}{r}} u\|_{L^r(\Omega)} \|\nabla u\|_{L^{\frac{2r}{r-2}}(\Omega)} \\ & \leq C \|\rho^{\frac{1}{r}} u\|_{L^r(\Omega)} (\|\nabla w\|_{L^{\frac{2r}{r-2}}(\Omega)} + \|\nabla v\|_{L^{\frac{2r}{r-2}}(\Omega)}) \\ & \leq C_{\epsilon} \|\nabla w\|_{L^2(\Omega)} + \epsilon \|\nabla^2 w\|_{L^2(\Omega)} + C. \end{aligned}$$

Here we use the interpolation inequality

$$\|f\|_{L^q(\Omega)} \leq C_{\epsilon} \|f\|_{L^2(\Omega)} + \epsilon \|\nabla f\|_{L^2(\Omega)}, \quad 2 \leq q < 6.$$

We infer from Proposition 2.1 that

$$\begin{aligned} & \|\sqrt{\rho} L^{-1} \nabla \operatorname{div}[p(\rho)u]\|_{L^2(\Omega)} \leq C \|p(\rho)u\|_{L^2(\Omega)} \leq C \|\sqrt{\rho} u\|_{L^2(\Omega)} \leq C, \\ & \|\sqrt{\rho} L^{-1} \nabla(\rho p' - p) \operatorname{div} u\|_{L^2(\Omega)} \\ & \leq \|\sqrt{\rho}\|_{L^3(\Omega)} \|L^{-1} \nabla(\rho p' - p) \operatorname{div} u\|_{L^6(\Omega)} \\ & \leq C \|\nabla L^{-1} \nabla(\rho p' - p) \operatorname{div} u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

Consequently, for $\epsilon > 0$ to be determined later,

$$(3.9) \quad \|\sqrt{\rho} F\|_{L^2(\Omega)}^2 \leq \epsilon \|\nabla^2 w\|_{L^2(\Omega)}^2 + C_{\epsilon} (1 + \|\nabla w\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2).$$

Noting that $Lw = \rho \partial_t w - \rho F$, we get by using Proposition 2.1 again that

$$\|\nabla^2 w\|_{L^2(\Omega)}^2 \leq C(\|\rho \partial_t w\|_{L^2(\Omega)}^2 + \|\rho F\|_{L^2(\Omega)}^2) \leq C(\|\sqrt{\rho} \partial_t w\|_{L^2(\Omega)}^2 + \|\sqrt{\rho} F\|_{L^2(\Omega)}^2),$$

which implies by taking $\epsilon = \frac{1}{3C}$ in (3.9) that

$$\|\sqrt{\rho} F\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|\sqrt{\rho} \partial_t w\|_{L^2(\Omega)}^2 + C(1 + \|\nabla w\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2).$$

Substituting this estimate into (3.8) and noting that $\|\nabla u(t)\|_{L^2(\Omega)}^2 \in L^1(0, T)$, the estimate (3.6) follows from Gronwall's inequality. \square

Corollary 3.3. Under the assumption (3.2), we have

$$\|\nabla u\|_{L^\infty(0,T;L^2(\Omega))}, \|u\|_{L^\infty(0,T;L^6(\Omega))}, \|\nabla u\|_{L^2(0,T;L^q(\Omega))} \leq C,$$

for any $q \in [2, 6]$.

Proof. This can be deduced from Proposition 3.2, (3.5) and Sobolev embedding theorem. \square

4. HIGH ORDER A PRIORI ESTIMATES FOR THE EFFECTIVE VISCOUS FLUX

In this section, we will give high order regularity estimates for w . This is possible if the initial data (ρ_0, u_0) satisfies the compatibility condition (1.5). We still assume that (ρ, u) is a strong solution of (1.1) in $[0, T)$ and satisfies (3.2). The energy estimates in this section are motivated by the calculations of D. Hoff [14].

We begin by introducing some notations. For a function or vector field (or even a 3×3 matrix) $f(t, x)$, the material derivative \dot{f} is defined by

$$\dot{f} \stackrel{\text{def}}{=} f_t + u \cdot \nabla f,$$

and $\text{div}(f \otimes u) \stackrel{\text{def}}{=} \sum_{j=1}^3 \partial_j(fu_j)$. For two matrices $A = (a_{ij})_{3 \times 3}$ and $B = (b_{ij})_{3 \times 3}$, we use the notation $A : B = \sum_{i,j=1}^3 a_{ij}b_{ij}$ and AB is as usual the multiplication of matrix.

We rewrite the second equation of (1.1) as

$$\rho \dot{u} + \nabla p(\rho) - Lu = 0.$$

By taking the material derivative to the above equation and using the fact $\dot{f} = f_t + \text{div}(fu) - f \text{div} u$, we obtain

$$\begin{aligned} & \rho \dot{u}_t + \rho u \cdot \nabla \dot{u} + \nabla p_t + \text{div}(\nabla p \otimes u) \\ (4.1) \quad & = \mu[\Delta u_t + \text{div}(\Delta u \otimes u)] + (\lambda + \mu)[\nabla \text{div} u_t + \text{div}((\nabla \text{div} u) \otimes u)]. \end{aligned}$$

Multiplying (4.1) by \dot{u} and integrating on Ω to obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |\dot{u}|^2 dx - \mu \int_{\Omega} \dot{u} \cdot (\Delta u_t + \text{div}(\Delta u \otimes u)) dx \\ & \quad - (\lambda + \mu) \int_{\Omega} \dot{u} \cdot ((\nabla \text{div} u_t) + \text{div}((\nabla \text{div} u) \otimes u)) dx \\ (4.2) \quad & = \int_{\Omega} p_t \text{div} \dot{u} + (\dot{u} \cdot \nabla u) \cdot \nabla p dx. \end{aligned}$$

The μ -term can be calculated as follows.

$$\begin{aligned}
& - \int_{\Omega} \dot{u} \cdot (\Delta u_t + \operatorname{div}(\Delta u \otimes u)) dx = \int_{\Omega} [\nabla \dot{u} : \nabla u_t + u \otimes \Delta u : \nabla \dot{u}] dx \\
& = \int_{\Omega} [|\nabla \dot{u}|^2 - \nabla(u \cdot \nabla u) : \nabla \dot{u} + u \otimes \Delta u : \nabla \dot{u}] dx \\
& = \int_{\Omega} [|\nabla \dot{u}|^2 - ((\nabla u \nabla u) + (u \cdot \nabla) \nabla u) : \nabla \dot{u} - \nabla(u \cdot \nabla \dot{u}) : \nabla u] dx \\
& = \int_{\Omega} [|\nabla \dot{u}|^2 - (\nabla u \nabla u) : \nabla \dot{u} - \operatorname{div}(\nabla u \otimes u) : \nabla \dot{u} \\
& \quad - (\nabla u \nabla \dot{u}) : \nabla u - ((u \cdot \nabla) \nabla \dot{u}) : \nabla u] dx \\
& = \int_{\Omega} [|\nabla \dot{u}|^2 - (\nabla u \nabla u) : \nabla \dot{u} + ((u \cdot \nabla) \nabla \dot{u}) : \nabla u \\
& \quad - (\nabla u \nabla \dot{u}) : \nabla u - ((u \cdot \nabla) \nabla \dot{u}) : \nabla u] dx \\
& \geq \int_{\Omega} \left[\frac{3}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \right] dx.
\end{aligned}$$

To estimate the $(\lambda + \mu)$ -term of (4.2), note that

$$\begin{aligned}
& \operatorname{div}((\nabla \operatorname{div} u) \otimes u) = \nabla(u \cdot \nabla \operatorname{div} u) - \operatorname{div}(\operatorname{div} u \nabla \otimes u) + \nabla(\operatorname{div} u)^2, \\
& \operatorname{div} \dot{u} = \operatorname{div} u_t + \operatorname{div}(u \cdot \nabla u) = \operatorname{div} u_t + u \cdot \nabla \operatorname{div} u + \nabla u : (\nabla u)'.
\end{aligned}$$

Here A' means the transpose of matrix A . We have

$$\begin{aligned}
& - \int_{\Omega} \dot{u} \cdot [\nabla \operatorname{div} u_t + \operatorname{div}((\nabla \operatorname{div} u) \otimes u)] dx \\
& = \int_{\Omega} [\operatorname{div} \dot{u} \operatorname{div} u_t + \operatorname{div} \dot{u}(u \cdot \nabla \operatorname{div} u) \\
& \quad - \operatorname{div} u(\nabla \dot{u})' : \nabla u + \operatorname{div} \dot{u}(\operatorname{div} u)^2] dx \\
& = \int_{\Omega} [|\operatorname{div} \dot{u}|^2 - \operatorname{div} \dot{u} \nabla u : (\nabla u)' - \operatorname{div} u(\nabla \dot{u})' : \nabla u + \operatorname{div} \dot{u}(\operatorname{div} u)^2] dx \\
& \geq \int_{\Omega} \left[\frac{1}{2} |\operatorname{div} \dot{u}|^2 - \frac{1}{4} |\nabla \dot{u}|^2 - C |\nabla u|^4 \right] dx.
\end{aligned}$$

We continue to estimate the pressure term.

$$\begin{aligned}
& \int_{\Omega} p_t \operatorname{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla p dx \\
&= \int_{\Omega} p'(\rho) \rho_t \operatorname{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla p dx \\
&= \int_{\Omega} -\rho p'(\rho) \operatorname{div} u \operatorname{div} \dot{u} - (u \cdot \nabla p(\rho)) \operatorname{div} \dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla p dx \\
&= \int_{\Omega} -\rho p'(\rho) \operatorname{div} u \operatorname{div} \dot{u} + p \left[\operatorname{div}((\operatorname{div} \dot{u})u) - \operatorname{div}((u \cdot \nabla \dot{u})) \right] dx \\
&= \int_{\Omega} -\rho p'(\rho) \operatorname{div} u \operatorname{div} \dot{u} + p \left[\operatorname{div} u \operatorname{div} \dot{u} - (\nabla u)' : \nabla \dot{u} \right] dx \\
&\leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla \dot{u}\|_{L^2(\Omega)} \leq C \|\nabla \dot{u}\|_{L^2(\Omega)},
\end{aligned}$$

where we used the assumption (3.2) and Corollary 3.3 in the last two inequalities.

Substituting those estimates into (4.2) yields

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \rho |\dot{u}|^2 dx + \mu \int_{\Omega} |\nabla \dot{u}|^2 dx + (\lambda + \mu) \int_{\Omega} |\operatorname{div} \dot{u}|^2 dx \\
(4.3) \quad & \leq C \int_{\Omega} |\nabla u|^4 dx + C \|\nabla \dot{u}\|_{L^2(\Omega)}.
\end{aligned}$$

To conclude the estimate by Gronwall's inequality, we will use the term $\|\sqrt{\rho} \dot{u}\|_{L^2(\Omega)}$ to control $\|\nabla u\|_{L^4(\Omega)}$. Thanks to the definition of w , we know that w satisfies

$$(4.4) \quad \mu \Delta w + (\lambda + \mu) \nabla \operatorname{div} w = \rho \dot{u} \text{ in } \Omega,$$

with the zero boundary condition. We get by Proposition 2.1 that

$$\|\nabla^2 w\|_{L^2(\Omega)} \leq C \|\rho \dot{u}\|_{L^2(\Omega)} \leq C \|\sqrt{\rho} \dot{u}\|_{L^2(\Omega)},$$

which together with the interpolation inequality, Corollary 3.3, and Proposition 2.1 leads to

$$\begin{aligned}
\|\nabla u\|_{L^4(\Omega)}^4 &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^6(\Omega)}^3 \leq C \|\nabla u\|_{L^6(\Omega)} \|\nabla u\|_{L^6(\Omega)}^2 \\
&\leq C \|\nabla u\|_{L^6(\Omega)}^2 (\|\nabla w\|_{L^6(\Omega)} + \|\nabla v\|_{L^6(\Omega)}) \\
&\leq C \|\nabla u\|_{L^6(\Omega)}^2 (1 + \|\nabla^2 w\|_{L^2(\Omega)}) \\
&\leq C \|\nabla u\|_{L^6(\Omega)}^2 (1 + \|\sqrt{\rho} \dot{u}\|_{L^2(\Omega)}).
\end{aligned}$$

Substituting this estimate into (4.3) and noting that $\|\nabla u(t)\|_{L^6(\Omega)}^2 \in L^1(0, T)$ by Corollary 3.3, we get by Gronwall's inequality that

$$(4.5) \quad \int_{\Omega} \rho |\dot{u}|^2 dx + \int_0^T \int_{\Omega} |\nabla \dot{u}|^2 dx dt \leq C,$$

with C depending only on T, M and ρ_0, u_0, g . Here we used the compatibility condition (1.5).

With the help of Sobolev embedding theorem and using the equation (4.4) again, we deduce from (4.5) that

Proposition 4.1. *Under the assumption (3.2), we have for all $2 \leq q \leq 6$,*

$$(4.6) \quad \|\nabla w\|_{L^2(0,T;L^\infty(\Omega))}, \|\nabla^2 w\|_{L^2(0,T;L^q(\Omega))} \leq C,$$

with the constant C depending on q, M, T and ρ_0, u_0, g .

5. PROOF OF THEOREM 1.3

Now we are in position to prove Theorem 1.3. We will prove it by the contradiction argument. Assume that $T^* < \infty$ and

$$\sup_{s \in [0, T^*)} \|\rho(s)\|_{L^\infty(\Omega)} < \infty.$$

By Theorem 1.1, it suffices to show that

$$(5.1) \quad \sup_{s \in [0, T^*)} \|\nabla \rho(s)\|_{L^q(\Omega)} < \infty.$$

Taking the derivative with respect to x for the first equation of (1.1) to obtain

$$(5.2) \quad \partial_t \nabla \rho + (u \cdot \nabla) \nabla \rho + \nabla u \nabla \rho + \operatorname{div} u \nabla \rho + \rho \nabla \operatorname{div} u = 0.$$

In the following estimates we will use

$$(5.3) \quad \begin{aligned} \|\nabla^2 v\|_{L^q(\Omega)} &\leq C \|\nabla \rho\|_{L^q(\Omega)}, \\ \|\nabla v\|_{L^\infty(\Omega)} &\leq C \left(1 + \|\nabla v\|_{BMO(\Omega)} \ln(e + \|\nabla^2 v\|_{L^q(\Omega)}) \right) \\ &\leq C \left(1 + \|\rho\|_{L^\infty \cap L^2(\Omega)} \ln(e + \|\nabla \rho\|_{L^q(\Omega)}) \right) \end{aligned}$$

$$(5.4) \quad \leq C \left(1 + \ln(e + \|\nabla \rho\|_{L^q(\Omega)}) \right)$$

with the second estimate followed from Proposition 2.1, 2.2 and Lemma 2.3.

Multiplying (5.2) by $q|\nabla\rho|^{q-2}\nabla\rho$ and integrating the resulting equation on Ω , we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla\rho|^q dx &\leq C \int_{\Omega} |\nabla u| |\nabla\rho|^q dx + q \int_{\Omega} \rho |\nabla \operatorname{div} u| |\nabla\rho|^{q-1} dx \\ &\leq C \|\nabla u\|_{L^\infty(\Omega)} \|\nabla\rho\|_{L^q(\Omega)}^q + C \|\nabla^2 u\|_{L^q(\Omega)} \|\nabla\rho\|_{L^q(\Omega)}^{q-1} \\ &\leq C(\|\nabla w\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}) \|\nabla\rho\|_{L^q(\Omega)}^q \\ &\quad + C(\|\nabla^2 w\|_{L^q(\Omega)} + \|\nabla^2 v\|_{L^q(\Omega)}) \|\nabla\rho\|_{L^q(\Omega)}^{q-1}, \end{aligned}$$

from which and (5.3)-(5.4), we infer that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla\rho|^q dx &\leq C(1 + \|\nabla v\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)}) \|\nabla\rho\|_{L^q(\Omega)}^q \\ &\quad + C\|\nabla^2 w\|_{L^q(\Omega)} \|\nabla\rho\|_{L^q(\Omega)}^{q-1} \\ &\leq C(1 + \|\nabla w\|_{W^{1,q}(\Omega)} + \ln(e + \|\nabla\rho\|_{L^q(\Omega)})) \|\nabla\rho\|_{L^q(\Omega)}^q \\ &\quad + \|\nabla^2 w\|_{L^q(\Omega)} \|\nabla\rho\|_{L^q(\Omega)}^{q-1}. \end{aligned}$$

Note that $\|\nabla w\|_{W^{1,q}(\Omega)} \in L^2(0, T^*)$ by Proposition 4.1. Then by Gronwall's inequality, we conclude the proof of (5.1) and hence Theorem 1.3. \square

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